DISCRETE FRAMES ON FINITE DIMENSIONAL QUATERNION HILBERT SPACES

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Abstract. A general theory of frames on finite dimensional quaternion Hilbert spaces is demonstrated along the lines of their complex counterpart. In this paper we established the frame decomposition theorem for quaternion Hilbert space. At the conclusion a perceptive clarification of why frames are important in signal transmission is given.

Key words: Frames, quaternions, quaternion Hilbert spaces.

1. INTRODUCTION

Frames were first introduced by Duffin and Schaeffer in a study of non-harmonic Fourier series [1]. However, among many others, the pioneering works of Daubechies et al. brought the proper attention to frames [2, 3]. Wavelets and coherent states of quantum optics are specific classes of continuous frames [4]. The study of frames has exploded in recent years, partly be- cause of their applications in digital signal processing [5, 6] and other areas of physical and engineering problems. In particular, they are an integral part of time frequency analysis. In this note we are primarily interested in frames on finite dimensional quaternion Hilbert spaces. There has been a constant surge in finding finite tight frames, largely as a result of several important applications such as internet coding, wireless communication, quantum detection theory, and many more [7, 8, 9, 5, 10]. It is crucial to find a specific class of frame to fit to a specific physical problem, because there is no universal class of frame that fit to all problems. As technology advances, physicists and engineers will face new problems and thereby our search for tools to solve them will continue. A Separable Hilbert space possesses an orthonormal basis and each vector in the Hilbert space can be uniquely written in terms of this orthonormal basis. Despite orthonormal bases are hard to find, this uniqueness restricted flexibility in applications and pleaded for an alternative. As a result frames entered to replace orthonormal bases. Frames are classes of vectors in Hilbert spaces. In a finite dimensional Hilbert space a typical frame possesses more vectors than the dimension of the space, and thereby each vector in the space can have infinitely many representations with respect to the frame. This redundancy of frames is the key to their success applications. The role of redundancy varies according to the requirements of the applications at hand. In fact, redundancy gives greater design flexibility which allows frames to be constructed to fit a particular problem in a manner not possible by a set of linearly independent vectors [4, 5, 2, 11]. Hilbert spaces can be defined over the fields$\mathbb{R}$, the set of all real numbers,$\mathbb{C}$ the set of all complex numbers, and H, the set of all quaternions only [12]. The fields $\mathbb{R}$ and $\mathbb{C}$ are associative and commutative and the theory of functional analysis is a well formed theory over real and complex Hilbert spaces. But the quaternions form a non-commutative associative algebra and this feature highly restricted mathematicians to work out a well-formed theory of functional analysis on quaternion Hilbert spaces. Further, due to the non-commutativity there are two types of Hilbert spaces on quaternions, called right quaternion Hilbert space and left quaternion Hilbert space. In assisting the study of frames the functional analytic properties of the underlying Hilbert space are essential. In the sequel we shall prove the necessary functional analytic properties as needed. This paper is a short version of [13]. For an enhanced explanation we refer the reader to [13].

2. QUATERNION ALGEBRA

In this section we shall define quaternions and some of their properties as needed here. For one may consult [12, 14, 15].

2.1. Quaternions. Let H denote the field of quaternions. Its elements are of the form $\boldsymbol{q} = x_0 + x_1 i + x_2 j + x_3 k$, where x_0, x_1, $x_2$and $x_3$are real numbers, and i, j, k are imaginary units such that $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i$ and $ki = -ik = j$. The quaternionic conjugate of $\boldsymbol{q}$ is defined to be $\bar{\boldsymbol{q}} = x_0 - x_1 i - x_2 j - x_3 k$. Quaternions can also be represented by using 2×2 complex matrices. It can be written as the linear combination of the matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$-i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},, \qquad i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

where σ_1, σ_2 and $\sigma_3$are the usual Pauli matrices. In this notations the quaternions can be written as (2.1) $q = x_0\sigma_0 + i\, x \,.\sigma$

with $x_0 \in \mathbb{R}, x = (x_1, x_2, x_3) \in \mathbb{R}^3$and $\sigma = (\sigma_1, -\sigma_2, \sigma_3)$. The quaternionic imaginary units are identified as, $i = \sqrt{-1}\sigma_1$, $j = \sqrt{-1}\sigma_2$, $k = \sqrt{-1}\sigma_3$. Thereby $q = x_0\sigma_0 + ix_1\sigma_1 - ix_2\sigma_2 + ix_3\sigma_3$and

(2.2) $$q = \begin{pmatrix} x_0 + ix_3 & -x_2 + ix_1 \\ x_2 + ix_1 & x_0 - ix_3 \end{pmatrix}$$

and $\bar{q} = q^\dagger$(matrix adjoint). Introducing the polar coordinates:

$$x_0 = r\cos\theta$$

$$x_1 = r\sin\theta\sin\varphi\cos\psi$$

$$x_2 = r\sin\theta\sin\varphi\sin\psi$$

$$x_3 = r\sin\theta\sin\varphi\cos\psi$$

where $r \in [0, \infty), \theta, \varphi \in [0, \pi]$, and $\psi \in [0, 2\pi)$, we may write

(2.3) $q = A(r)e^{i\theta}\sigma(\tilde{n})$

where $A(r) = r\sigma_0$ and

(2.4) $$\sigma(\tilde{n}) = \begin{pmatrix} \cos\varphi & \sin\varphi e^{i\psi} \\ \sin\varphi e^{-i\psi} & -\cos\varphi \end{pmatrix}$$

The matrices $A(r)$ and $\sigma(\tilde{n})$ satisfy the conditions,

(2.5) $A(r) = A(r)^\dagger, \sigma(\tilde{n})^2 = \sigma_0$,

with

(2.6) $\sigma(\tilde{n})^{\dagger} = \sigma(\tilde{n}),\ [A(r), \sigma(\tilde{n})] = 0,$

where $[A, B] = AB - BA$ is called Lie bracket.

Note that $|\boldsymbol{q}|^2 := \bar{q}q = r^2\ \sigma_0$ .Thereby $|\boldsymbol{q}|^2 = (x_0^2 + x_1^2 + x_2^2 + x_3^2)\mathbb{I}_2$defines a real norm on H and $\mathbb{I}_2$ stands for the 2×2 identity matrix.

2.2. Properties of Quaternions: The quaternion product allows the following properties. $For\ q, r, s \in H$, we have

(a) $q(rs) = (qr)s$ (associative)

(b) $q(r + s) = qr + qs.$

(c) For each $q \neq 0$, there exists r such that $qr = 1$

(d) If $qr = qs$ then $r = s$ whenever $q \neq 0$

The quaternion product is not commutative

3. FRAMES IN QUATERNION HILBERT SPACE

Definition 3.1. Let V_H^L is a vector space under left multiplication by quaternionic scalars, where H stands for the quaternion algebra. For $f, g, h \in V_H^L$ and $q \in H$, the inner product

$\langle . | . \rangle : V_H^L \times V_H^L \to H$satisfies the following properties:

(a) $\overline{\langle f|g\rangle} = \langle g|f\rangle$

(b) $\|f\|^2 = \langle f|f\rangle > 0$ unless $f = 0$, a real norm

(c) $\langle f\ |g + h\rangle = \langle f\ |g\ \rangle + \langle f\ |\ h\rangle$

(d) $\langle \boldsymbol{q}f|g\rangle = \boldsymbol{q}\langle f|g\rangle$

(e) $\langle f|\boldsymbol{q}g\rangle = \langle f|g\rangle\bar{\boldsymbol{q}}$.

Note that the space $V_H^L$is together with $\langle . | . \rangle$is a separable Hilbert space. Properties of left quaternion Hilbert spaces as needed here can be listed as follows: For $f, g, \in V_H^L$ and $p, q \in H$, we have

(a) $pf + qg \in V_H^L$

(b) $p(f + g) = pf + pg$

(c) $(pq)f = p(qf)$

(d) $(p + q)f = pf + qf.$

Proposition 3.2. [12] (Schwartz inequality) $\langle f|g\rangle\langle g|f\rangle \leq \|f\|^2\ \|g\|^2,$ for all $f, g, \in V_H^L$

Definition 3.3. (*Basis*) Let V_H^Lbe a finite dimensional left quaternion Hilbert space, equipped with an inner product $\langle . | . \rangle$ which wechoose to be linear in the second entry. A sequence $\{e_k\}_{k=1}^m$ is a basis for V_H^L if the following two conditions are satisfied:

(1) $V_H^L = left\ span\{e_k\}_{k=1}^m$

(2) $\{e_k\}_{k=1}^m$is a linearly independent set.

We now introduce the frames on finite dimensional left quaternion Hilbert spaces. We shall show that the complex treatment adapt to the quaternions as well. In this paper *left span* means left span over the quaternion scalar field, H. We shall also prove the functional analytic properties for quaternions as needed here, and these proofs are the adaptation of the proofs of the complex cases given in [16]. The theory of frames offered here, more or less, follows the lines of [5].

Definition 3.4. (*Frames***)** A countable family of elements $\{f_k\}_{k\in I}$ in $V_H^L$is a frame for V_H^L if there exist constants $A, B > 0$such that

(3.1) $A\|f\|^2 \leq \sum_{k\in I}|\langle f|f_k\rangle|^2 \leq B\|f\|^2$,

for all $f \in V_H^L$.

The numbers A and B are called frame bounds. They are not unique. The *optimal lower frame bound* is the supremum over all lower frame bounds, and the *optimal upper frame bound* is the infimum over all upper frame bounds. Note that the optimal frame bounds are actually the frame bounds. A frame is said to be normalized if$\|f_k\| = 1$, for all $k \in I$. In this note we shall only consider finite frames $\{f_k\}_{k=1}^{m}, m \in \mathbb{N}$. With this restriction, Schwartz inequality shows that

(3.2) $\sum_{k=1}^{m} |\langle f | f_k \rangle|^2 \le \sum_{k=1}^{m} \|f_k\|^2 \; \|f\|^2$,

for all $f \in V_H^L$.From (3.2) it is clear that the upper frame condition is always satisfied with $A = \sum_{k=1}^{m} \|f_k\|^2$. In order for the lower condition in (3.1) to be satisfied, it is necessary that $left\ span\,\{f_k\}_{k=1}^{m} = V_H^L$.Let us see this in the following.

Proposition 3.5. Let $\{f_k\}_{k=1}^{m}$be a sequence in V_H^L.Then $\{f_k\}_{k=1}^{m}$is a frame for $left\ span\,\{f_k\}_{k=1}^{m}$.

Corollary 3.6. A family of elements $\{f_k\}_{k=1}^{m}$in $V_H^L$is a frame for V_H^L if and only if $leftspan\,\{f_k\}_{k=1}^{m} = V_H^L$.

Proof. Suppose that $\{f\}_{k=1}^{m}$ is a frame for V_H^L.Then there exist $A, B > 0$ such that

(3.3) $A\|f\|^2 \le \sum_{k=1}^{m} |\langle f | f_k \rangle|^2 \le B\|f\|^2$

for all $f \in V_H^L$. If there exists $f \in V_H^L$ such that $f \notin leftspan\{f_k\}_{k=0}^{\infty}$. Then $f \ne \sum_{k=1}^{m} c_k\, f_k$ for all sequences $\{c_k\}_{k=1}^{m} \subset H$. That is,$\|f\|^2 \ne \sum_{k=1}^{m} |c_k|^2 \|f_k\|^2$ for any sequence $\{c_k\}_{k=1}^{m} \subset H$. Set $c_k = \left\langle \frac{f}{\sqrt{B}} \middle| \frac{f_k}{\|f_k\|} \right\rangle \in H$ for all $k = 1, 2, \cdots, m$.

Thereby

$$\|f\|^2 \ne \sum_{k=1}^{m} \left| \left\langle \frac{f}{\sqrt{B}} \middle| \frac{f_k}{\|f_k\|} \right\rangle \right|^2 \|f_k\|^2$$

$$= \frac{1}{B} \sum_{k=1}^{m} |\langle f | f_k \rangle|^2$$

$$\le \frac{1}{B} B\|f\|^2 = \|f\|^2 \text{by} \qquad (3.3),$$

which is a contradiction. Thereby $V_H^L \subseteq leftspan\{f_k\}_{k=1}^{m}$. Clearly $leftspan\{f_k\}_{k=1}^{m} \subseteq V_H^L$. Thereby the conclusion follows.

Conversely suppose that $leftspan\{f_k\}_{k=1}^{m} = V_H^L$. From proposition (3.5) $\{f_k\}_{k=1}^{m}$is a frame for $eftspan\{f_k\}_{k=1}^{m}$, thereby $\{f_k\}_{k=1}^{m}$is a frame forV_H^L. □

From the above corollary it is clear that a frame is an over complete family of vectors in a finite dimensional Hilbert space.

3.1. Frame operator in left quaternion Hilbert space.

3.1.1. *Operators on left quaternion Hilbert spaces.* Let $O : V_H^L \longrightarrow V_H^L$be a quaternion linear operator. In this case, the operators always act from the left as $O|f\rangle$ and the scalar multiple of the operator is taken from the left as qO. Further the operators obey the following rules:

(i) $O|\boldsymbol{q}f\rangle = \boldsymbol{q}(O|f\rangle)$.

(ii)$\langle f|Og\rangle = \langle O^{\dagger}f|g\rangle$; $O^{\dagger}$is the adjoint of O.

(iii) $(\boldsymbol{q}O)|f\rangle = O|\boldsymbol{q}f\rangle$.

For a detail explanation we refer the reader to [12].

Definition 3.7. [12] Let V_H^L be any leftquarternion Hilbert space. A mapping $S : V_H^L \longrightarrow V_H^L$is said to be linear if, $S(\alpha f + \beta g) = \alpha S(f) + \beta S(g)$;for all $f, g \in V_H^L$ and $\alpha, \beta \in H$.

Definition 3.8. [12] A linear operator $S : V_H^L \longrightarrow V_H^L$is said to be bounded if, $\|Sf\| \le Kf$,forsome constant $K > 0$ and all $\in V_H^L$.

Definition 3.9. [12](*Adjoint operator*)
Let $S : V_H^L \longrightarrow V_H^L$be a bounded linear operator on a left quaternion Hilbert space. We define its adjoint to be the operator $S^{\dagger} : V_H^L \longrightarrow V_H^L$thathas the property
(3.4) $\langle f|Sg\rangle = \langle S^{\dagger}f|g\rangle$;
for all $f, g \in V_H^L$.

Definition 3.10. [12](*Self-adjoint operator*)
Let V_H^L be a left quaternion Hilbert space. A

bounded linear operator S on V_H^L is called *self-adjoint*, if $S^\dagger = S$.
Analogous to commutative case, we now provide elementary results that hold for left quaternion space, which is non commutative.

Lemma 3.11. Let U_H^L,V_H^Lbe finite dimensional left quaternion Hilbert spaces and $S : U_H^L \longrightarrow V_H^L$ be a linear mapping then

$$dimR_S + dimN_S = dimU_L^H$$

where $R_S := image\ of\ S, N_S := ker\ S.$

Lemma 3.12. Let $S : U_H^L \longrightarrow V_H^L$be a linear mapping. S is one to one if and only if $N_S = \{0\}$.

Lemma 3.13. Let U_H^L,$V_H^L$are finite dimensional left quaternion Hilbert spaces with samedimension. Let $S : U_H^L \longrightarrow V_H^L$be a linearmapping. If S is one to one then S is onto.

Lemma 3.14. (*Pythagoras' law*) Suppose that f and g is an arbitrary pair of orthogonal vectors in the left quaternion Hilbert space V_H^L.Then we have Pythagoras' formula

(3.5) $$\|f + g\|^2 = \|f\|^2 + \|g\|^2 .$$

Lemma 3.15. Let $T : H^m \longrightarrow V_H^L$ be a linear mapping and $T^\dagger : V_H^L \longrightarrow H^m$be its adjointoperator. Then $N_T = R_{T^\dagger}^\perp$, where $N_T := ker\ T$and $R_{T^\dagger} := range\ of\ T^\dagger$.
For detail proofs of the above lemmas we refer the reader to [13].

3.1.2. Frame operators. Consider now a left quaternion Hilbert space, V_H^L with a frame $\{f_k\}_{k=1}^m$ and define a linear mapping $T : H^m \longrightarrow V_H^L$ by

(3.6) $$T\ \{c_k\}_{k=1}^m = \sum_{k=1}^m c_k f_k,\quad c_k \in H.$$

T is usually called the *pre-frame operator or the synthesis operator.* The adjoint operator $T^\dagger : V_H^L \longrightarrow H^m$, by

(3.7) $$T^\dagger f = \{\langle f|f_k\rangle\}_{k=1}^m$$

is called the analysis operator. By composing T with its adjoint we obtain the frame operator $S : V_H^L \longrightarrow V_H^L$by

(3.8) $$Sf = TT^\dagger f = \sum_{k=1}^m \langle f|f_k\rangle f_k$$

Note that in terms of the frame operator, for $f \in V_H^L$

$$\begin{aligned}\langle Sf|f\rangle &= \langle \textstyle\sum_{k=1}^m \langle f|f_k\rangle f_k|f\rangle \\ &= \sum_{k=1}^m \langle f|f_k\rangle\langle f_k|f\rangle \\ &= \sum_{k=1}^m |\langle f|f_k\rangle|^2.\end{aligned}$$

Thereby

(3.9) $\langle Sf|f\rangle = \sum_{k=1}^m |\langle f|f_k\rangle|^2 , f \in V_H^L$.

A frame $\{f_k\}_{k=1}^m$ is tight if we can choose $A = B$ in the definition (3.4), hence (3.1) gives $\sum_{k=1}^m |\langle f|f_k\rangle|^2 = A\|f\|^2$,for all $f \in V_H^L$.

Thereby $\langle Sf|f\rangle = A\|f\|^2$,for all $f \in V_H^L$.

Theorem 3.16. *(Frame decomposition theorem)Let $\{f_k\}_{k=1}^m$be a frame for $V_H^L$with frame operatorS. Then*
(1) S is self-adjoint and invertible.
(2) Every $f \in V_H^L$ can be represented as

$$f = \sum_{k=1}^m \langle f|S^{-1}f_k\rangle f_k = \sum_{k=1}^m \langle f|f_k\rangle S^{-1} f_k$$

(3) *If $f \in V_H^L$has the representation $f = \sum_{k=1}^m c_k f_k$ for some scalar coefficients $\{c_k\}_{k=1}^m$then*

$$\sum_{k=1}^{m}|c_k|^2 = \sum_{k=1}^{m}|\langle f|S^{-1}f_k\rangle|^2 + \sum_{k=1}^{m}|c_k - \langle f|S^{-1}f_k\rangle|^2$$

Proof. (1) $S : V_H^L \longrightarrow V_H^L$ by $Sf = TT^\dagger f = \sum_{k=1}^{m}\langle f|f_k\rangle f_k$, for all $f \in V_H^L$.
Now

$$S^\dagger = (TT^\dagger)^\dagger = (T^\dagger)^\dagger T^\dagger = TT^\dagger = S.$$

It follows that S is self-adjoint. We have $ker\, S = \{f : Sf = 0\}$. Let $f \in ker\, S$, then $Sf = 0$. Therefore

$$0 = \langle Sf|f\rangle = \langle \textstyle\sum_{k=1}^{m}\langle f|f_k\rangle f_k|f\rangle = \sum_{k=1}^{m}|\langle f|f_k\rangle|^2.$$

Thereby $\sum_{k=1}^{m}|\langle f|f_k\rangle|^2 = 0$. Since $\{f_k\}_{k=1}^{m}$ be a frame for V_H^L by definition (3.4),

$$A\|f\|^2 \le \sum_{k=1}^{m}|\langle f|f_k\rangle|^2 \le B\|f\|^2,$$

for all $f \in V_H^L$.
Hence $A\|f\|^2 \le 0 \le B\|f\|^2$, for all $f \in V_H^L$, and $A, B > 0$. So $\|f\|^2 = 0$. Thereby $f = 0$, for all $f \in V_H^L$, it follows that $N_S = \{0\}$. Hence S is one to one. Since V_H^L is of the finitedimension, from the lemma (3.13) S is onto. Therefore S is invertible.

(2) If S is self-adjoint then S^{-1} is self-adjoint.
For, Consider

$$(S^{-1})^\dagger = (S^\dagger)^{-1} = S^{-1}.$$

Thereby S^{-1} is self-adjoint. If the function $S : V_H^L \longrightarrow V_H^L$ is linear and bijection then S^{-1} is linear. For, since S is onto, $S^{-1} : V_H^L \longrightarrow V_H^L$.
Let $f, g \in V_H^L$ then there exists $k, h \in V_H^L$ such that $S^{-1}(f) = k$ and $S^{-1}(g) = h$. Thereby $f = S(k)$ and $g = S(h)$.
Let $\alpha, \beta \in H$, then

$$\begin{aligned} S^{-1}(\alpha f + \beta g) &= S^{-1}(\alpha S(k) + \beta S(h)) \\ &= S^{-1}(S(\alpha k + \beta h)) \\ &= \alpha k + \beta h \\ &= \alpha S^{-1}(f) + \beta S^{-1}(g) \end{aligned}$$

Thereby for all $f, g \in V_H^L$ and $\alpha, \beta \in H$,

$$S^{-1}(\alpha f + \beta g) = \alpha S^{-1}(f) + \beta S^{-1}(g).$$

Hence S^{-1} is linear.
Let $f, \in V_H^L$ then

$$\begin{aligned} f &= SS^{-1}f \\ &= TT^\dagger S^{-1}f \\ &= \textstyle\sum_{k=1}^{m}\langle S^{-1}f|f_k\rangle f_k \\ &= \textstyle\sum_{k=1}^{m}\langle f|(S^{-1})^\dagger f_k\rangle f_k \\ &= \textstyle\sum_{k=1}^{m}\langle f|S^{-1}f_k\rangle f_k \end{aligned}$$

as S^{-1} is self adjoint. Thereby for every $\in V_H^L$,

(3.10) $$f = \sum_{k=1}^{m}\langle f|S^{-1}f_k\rangle f_k$$

Similarly we have

$$\begin{aligned} f &= S^{-1}Sf \\ &= S^{-1}\,TT^\dagger f \\ &= S^{-1}(\textstyle\sum_{k=1}^{m}\langle f|f_k\rangle f_k) \\ &= \textstyle\sum_{k=1}^{m} S^{-1}(\langle f|f_k\rangle f_k) \\ &= \textstyle\sum_{k=1}^{m}\langle f|f_k\rangle S^{-1}f_k \end{aligned}$$

as S^{-1} is linear. Thereby for every $f \in V_H^L$

(3.11) $$f = \sum_{k=1}^{m}\langle f|f_k\rangle S^{-1}f_k$$

From (3.10) and (3.11), for every $f \in V_H^L$,

$$f = \sum_{k=1}^{m}\langle f|S^{-1}f_k\rangle f_k = \sum_{k=1}^{m}\langle f|f_k\rangle S^{-1}f_k$$

(3) Let $f \in V_H^L$, from corollary (3.6)

$f = \sum_{k=1}^{m} c_k f_k$, for some $c_k \in H$

From previous part,

(3.12) $f = \sum_{k=1}^{m} c_k f_k = \sum_{k=1}^{m} \langle f | S^{-1} f_k \rangle f_k$

Hence

(3.13) $\sum_{k=1}^{m} (c_k - \langle f | S^{-1} f_k \rangle) f_k = 0.$

Thereby $\sum_{k=1}^{m} d_k f_k = 0$, for some $d_k = c_k - \langle f | S^{-1} f_k \rangle \in H$.

From (3.6), pre-frame operator $T : H^m \longrightarrow V_H^L$, is defined by

(3.14) $T\{d_k\}_{k=1}^{m} = \sum_{k=1}^{m} d_k f_{k},$

where $\{d_k\}_{k=1}^{m} = \{c_k\}_{k=1}^{m} - \{\langle f | S^{-1} f_k \rangle\}_{k=1}^{m}$.

We have $N_T = \{\{d_k\}_{k=1}^{m} | T\{d_k\}_{k=1}^{m} = 0\}$, therefore $\{d_k\}_{k=1}^{m} \in N_T$.

From lemma (3.15) $N_T = R_{T^\dagger}^{\perp}$, then

(3.15) $\{c_k\}_{k=1}^{m} - \{\langle f | S^{-1} f_k \rangle\}_{k=1}^{m} \in R_{T^\dagger}^{\perp}$

From (3.6) and (3.7) we have $T^\dagger : V_H^L \longrightarrow H^m$ by $T^\dagger f = \{\langle f | f_k \rangle\}_{k=1}^{m}$ and $S: V_H^L \longrightarrow V_H^L$ by

$$Sf = TT^\dagger f = \sum_{k=1}^{m} \langle f | f_k \rangle f_k.$$

Hence $T^\dagger(S^{-1} f) = \{\langle S^{-1} f | f_k \rangle\}_{k=1}^{m}$.

Therefore

$$\{\langle S^{-1} f | f_k \rangle\}_{k=1}^{m} \in R_{T^\dagger}.$$

Since S^{-1} is self adjoint,

$$\{\langle S^{-1} f | f_k \rangle\}_{k=1}^{m} = \{\langle f | S^{-1} f_k \rangle\}_{k=1}^{m}.$$

Hence

(3.16) $\{\langle f | S^{-1} f_k \rangle\}_{k=1}^{m} \in R_{T^\dagger}$.

Now we can write,

$$\{c_k\}_{k=1}^{m} = \{c_k\}_{k=1}^{m} - \{\langle f | S^{-1} f_k \rangle\}_{k=1}^{m} + \{\langle f | S^{-1} f_k \rangle\}_{k=1}^{m}$$

From (3.15), (3.16) and lemma (3.14), it follows that

$$\sum_{k=1}^{m} |c_k|^2 = \sum_{k=1}^{m} |\langle f | S^{-1} f_k \rangle|^2 + \sum_{k=1}^{m} |c_k - \langle f | S^{-1} f_k \rangle|^2$$

□

Theorem (3.16) is one of the most important results about frames, and

$$f = \sum_{k=1}^{m} \langle f | S^{-1} f_k \rangle f_k = \sum_{k=1}^{m} \langle f | f_k \rangle S^{-1} f_k$$

is called the *frame decomposition.*

Note that if $\{f_k\}_{k=1}^{m}$ is a frame but not a basis,there exists non-zero sequences $\{g_k\}_{k=1}^{m}$ such that $\sum_{k=1}^{m} g_k f_k = 0$.Thereby $f \in V_H^L$ can be written as

$$f = \sum_{k=1}^{m} \langle f | S^{-1} f_k \rangle f_k + \sum_{k=1}^{m} g_k f_k = \sum_{k=1}^{m} (\langle f | S^{-1} f_k \rangle + g_k) f_k$$

showing that f has many representations as superposition of the frame elements.

Corollary 3.17. *Assume that $\{f_k\}_{k=1}^{m}$ is a basis for V_H^L.Then there exists a unique family $\{g_k\}_{k=1}^{m}$ in V_H^L such that*

(3.17) $f = \sum_{k=1}^{m} \langle f | g_k \rangle f_k,$

for all $f \in V_H^L$. In terms of the frame operator,$\{g_k\}_{k=1}^{m} = \{S^{-1} f_k\}_{k=1}^{m}$. Furthermore$\langle f_j | g_k \rangle = \delta_{j,k}$.

4. CONCLUSION

We conclude by giving a perceptive clarification of why frames are important in signal transmission. Let us assume that we want to transmit the signal f belonging to a left quaternion Hilbert space from a transmitter $\mathcal{A}$ to a receiver$\mathcal{R}$. If both $\mathcal{A}$ and $\mathcal{R}$ have knowledge of frame $\{f_k\}_{k=1}^m$ for V_H^L, this can be done if $\mathcal{A}$ transmits the frame coefficients $\{\langle f|S^{-1}f_k\rangle\}_{k=1}^m$; based on knowledge of these numbers, the receiver $\mathcal{R}$ can reconstruct the signal f using the frame decomposition. Now assume that $\mathcal{R}$ receives a noisy signal, meaning a perturbation $\{\langle f|S^{-1}f_k\rangle + c_k\}_{k=1}^m$ of the correct frame coefficients. Based on the received coefficients, $\mathcal{R}$ will assert that the transmitted signal was

$$\sum_{k=1}^{m}(\langle f|S^{-1}f_k\rangle + c_k)\, f_k$$

$$= \sum_{k=1}^{m}\langle f|S^{-1}f_k\rangle f_k + \sum_{k=1}^{m} c_k\, f_k$$

$$= f + \textstyle\sum_{k=1}^{m} c_k\, f_k.$$

this differs from the correct signal f by the noise $\sum_{k=1}^{m} c_k\, f_k$.Minimizing this noise for various signals with different types of noises has been a hot topic in signal processing. We shall touch this issue in the future. For now, if $\{f_k\}_{k=1}^m$ is over complete, parts of the noise contribution might add up to zero and cancel. This will never happen if $\{f_k\}_{k=1}^m$ is an orthonormal basis. In that case

$$\left\|\sum_{k=1}^{m} c_k\, f_k\right\|^2 = \sum_{k=1}^{m}|c_k|^2$$

so each noise contribution will make the reconstruction worse.

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