RIGHT QUATERNIONIC COHERENT STATES AND THE HEISENBERG UNCERTAINTY PRINCIPLE

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ABSTRACT. Parallel to the quantization of the complex plane, using the canonical coherent states of a right quaternionic Hilbert space, quaternion field of quaternionic quantum mechanics is quantized and using the quantization the position and momentum operators are obtained by us in [1]. In this article, we show that the right quaternionic canonical coherent states saturate the Heisenberg uncertainty relation and thereby they form a set of intelligent states and also we show that they are a set of minimum uncertainty states.

Key words: Quaternion, Quantization, Coherent states, Heisenberg uncertainty.

1. Introduction

Quantization is commonly understood as the transition from classical to quantum mechanics. One may also say, to a certain extent, quantization relates to a larger discipline than just restricting to specific do-mains of physics. In physics, the quantization is a procedure that associates with an algebra A_{cl} of classical observables an algebra A_q of quantum observables. The algebra A_q is usually realized as a commutative Poisson algebra of derivable functions on a symplectic (or phase) space$X$. The algebra A_q is, however, non-commutative in general and the quantization procedure must pro-vide a correspondence $A_{cl} \longmapsto A_q : f \longmapsto A_f$.

Most physical quantum theories may be obtained as the result of a canonical quantization procedure which simply replaces the classical variables by quantum observables. However, among the various quantization procedures available in the literature, the coherent state quantization (CS quantization) appear quite arbitrary because the only structure that a space X must possess is a measure. Once a family of CS or frame labeled by a measure space X is given one can quantize the measure space X. Various quantization schemes and their advantages and drawbacks are discussed in detail, for example, in [2, 3, 4, 5].

Due to the non commutativity of quaternions, quaternionic Hilbert spaces are formed by right or left multiplication of vectors by quaternionic scalars; the two different conventions give isomorphic versions of the theory. Quaternions can always be represented, through symplectic component functions, as a pair of complex numbers and thereby quaternions possess a symplectic structure. However, the quaternionic quantum mechanics is inequivalent to complex quantum mechanics. In analogy with the complex quantum mechanics (CQM), states of quaternionic quantum mechanics (QQM) are described by vectors of a separable quaternionic Hilbert space and observables in QQM are represented by quaternion linear and self-adjoint operators [6].

The CS quantization in the CQM is a well-known and well-studied problem. Using the method of CS quantization, various phase spaces such as complex field, complex unit that the states providing equality in the uncertainty relation do not, in general, reach minimum uncertainty. The uncertainty relation limits the precise knowledge of conjugate physical quantities of a system. The states which minimize the uncertainty relation can describe the quantum system as precisely as possible. First, for given two self-adjoint operators A and B, one can obtain, using the Cauchy-Schwartz inequality, the uncertainty relation,

$$\langle \Delta A\rangle\langle \Delta B\rangle \geq \frac{1}{2}|\langle [A,B]\rangle|, \tag{1.1}$$

where the variance is given by

$$\langle \Delta A\rangle = \left[\langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2\right]^{\frac{1}{2}}$$

and the expectation values are taken over the normalized states of the system. States for which equality is achieved in (1.1) are called intelligent states. States which also minimize the uncertainty product (1.1) are called minimum uncertainty states. For position and momentum observables Q and P, if the commutation relation is the multiple of the identity operator, $[Q,P] = iI$, the right hand side of (1.1) is state-independent. In this case, the intelligent states and the minimum uncertainty states coincide and in CQM these states are the canonical CS of the harmonic oscillator. However, in the general case, if $[A,B] = iC$, where C is an operator different from the identity operator, then intelligent states and the minimum uncertainty states are generally different. Intelligent states $|\lambda\rangle_{AB}$ for operators A and B are determined from the eigenvalue equation,

$$(A + i\gamma B)|\lambda\rangle_{AB} = \lambda|\lambda\rangle_{AB},$$

where λ is a complex eigenvalue and γ is a real parameter. For details we refer the reader to [10] and the references therein.
The operator properties of CQM do not translate directly to the operators of QQM [11]. In this article for QQM, using the CS quantization and the position and momentum observables obtained in [1], we show that the right quaternionic canonical CS saturate the Heisenberg uncertainty relation, and thereby they form a set of minimum uncertainty and intelligent states.

2. Mathematical preliminaries

In order to make the paper self-contained, we recall a few facts about quaternions which may not be well-known. In particular, we revisit the 2×2 complex matrix representations of quaternions, quaternionic Hilbert spaces. For further details we refer the deader to [6, 12, 11, 14, 16].

2.1. **Quaternions.** Let H denote the field of quaternions. Its elements are of the form $\mathbf{q} = x_0 + x_1 i + x_2 j + x_3 k$ where x_0, x_1, x_2 and x_3 are real numbers, and i, j, k are imaginary units such that $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. The quaternionic conjugate of $\mathbf{q}$ is defined to be $\overline{\mathbf{q}} = x_0 - x_1 i - x_2 j - x_3 k$. We shall find it convenient to use the representation of quaternions by 2×2 complex matrices:

$$\mathbf{q} = x_0\sigma_0 + i\underline{x}\cdot\underline{\sigma}, \tag{2.1}$$

with $x_0 \in \mathbb{R}$, $\underline{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\sigma_0 = \mathbb{I}_2$, the 2×2 identity matrix, and $\underline{\sigma} = (\sigma_1, -\sigma_2, \sigma_3)$, where the σ_ℓ, $\ell = 1, 2, 3$ are the usual Pauli matrices. The quaternionic imaginary units are identified as, $i = \sqrt{-1}\sigma_1$, $j = -\sqrt{-1}\sigma_2$, $k = \sqrt{-1}\sigma_3$. Thus,

$$\mathbf{q} = \begin{pmatrix} x_0 + ix_3 & -x_2 + ix_1 \\ x_2 + ix_1 & x_0 - ix_3 \end{pmatrix} \tag{2.2}$$

and $\overline{\mathbf{q}} = \mathbf{q}^\dagger$ (matrix adjoint) . Introducing the polar coordinates:

$$\begin{aligned} x_0 &= r\cos\theta, \\ x_1 &= r\sin\theta\sin\phi\cos\psi, \\ x_2 &= r\sin\theta\sin\phi\sin\psi, \\ x_3 &= r\sin\theta\cos\phi, \end{aligned}$$

where $(r, \phi, \theta, \psi) \in [0,\infty) \times [0,\pi] \times [0, 2\pi)^2$, we may write

$$\mathbf{q} = A(r)e^{i\theta\sigma(\widehat{n})}, \tag{2.3}$$

where

$$A(r) = r\sigma_0 \tag{2.4}$$

and

$$\sigma(\widehat{n}) = \begin{pmatrix} \cos\phi & \sin\phi e^{i\psi} \\ \sin\phi e^{-i\psi} & -\cos\phi \end{pmatrix}. \tag{2.5}$$

The matrices $A(r)$ and $\sigma(\widehat{n})$ satisfy the conditions,

$$A(r) = A(r)^\dagger,\ \sigma(\widehat{n})^2 = \sigma_0,\ \sigma(\widehat{n})^\dagger = \sigma(\widehat{n}) \tag{2.6}$$

and $[A(r), \sigma(\widehat{n})] = 0$. Note that a real norm on H is defined by

$$|\mathbf{q}|^2 := \overline{\mathbf{q}}\mathbf{q} = r^2\sigma_0 = (x_0^2 + x_1^2 + x_2^2 + x_3^2)\mathbb{I}_2.$$

A typical measure on H may take the form

$$d\varsigma(r,\theta,\phi,\psi) = d\tau(r)\, d\theta\, d\Omega(\phi,\psi) \tag{2.7}$$

with $d\Omega(\phi,\psi) = \dfrac{1}{4\pi}\sin\phi\, d\phi\, d\psi$. Note also that for $\mathbf{p},\mathbf{q} \in H$, we have $\overline{\mathbf{pq}} = \overline{\mathbf{q}}\ \overline{\mathbf{p}}$, $\mathbf{pq} \neq \mathbf{qp}$, $\mathbf{q}\overline{\mathbf{q}} = \overline{\mathbf{q}}\mathbf{q}$, and real numbers commute with quaternions. In defining the position and momentum operators, we shall also need the sliced version of quaternions. We borrow the materials as needed here from [13]. Let

$$\begin{aligned}\mathbb{S} &= \{\mathbf{q} = x_1 i + x_2 j + x_3 j \mid \\ &\quad x_1, x_2, x_3 \in \mathbb{R},\ x_1^2 + x_2^2 + x_3^2 = 1\},\end{aligned}$$

we call it a quaternion sphere. For any non-real quaternion $\mathbf{q} \in H \smallsetminus \mathbb{R}$, there exist, and are unique, $x, y \in \mathbb{R}$ with $y > 0$, and $I \in \mathbb{S}$ such that $\mathbf{q} = x + yI$. For every quaternion $I \in \mathbb{S}$, the complex line $L_I = \mathbb{R} + I\mathbb{R}$ passing through the origin, and containing 1 and I, is called a quaternion slice. Thereby, we can see that

$$H = \bigcup_{I\in\mathbb{S}} L_I \quad \text{and} \quad \bigcap_{I\in\mathbb{S}} L_I = \mathbb{R} \tag{2.8}$$

One can also easily see that $L_I \subset H$ is commutative, while, elements from two different quaternion slices, L_I and L_J (for $I, J \in \mathbb{S}$ with $I \neq J$), do not necessarily commute.

2.2. **Quaternionic Hilbert spaces.** In this subsection we define left and right quaternionic Hilbert spaces. For details we refer the reader to [6]. We also define the Hilbert space of square integrable functions on quaternions based on [17, 11, 14].

2.2.1. *Right Quaternionic Hilbert Space.* Let V_H^R be a linear vector space under right multiplication by quaternionic scalars (again H standing for the field of quaternions). For $f, g, h \in V_H^R$ and $\mathbf{q} \in H$, the inner product

$$\langle \cdot \mid \cdot \rangle : V_H^R \times V_H^R \longrightarrow H$$

satisfies the following properties

(i) $\overline{\langle f \mid g\rangle} = \langle g \mid f\rangle$
(ii) $\|f\|^2 = \langle f \mid f\rangle > 0$ unless $f = 0$, a real norm
(iii) $\langle f \mid g + h\rangle = \langle f \mid g\rangle + \langle f \mid h\rangle$
(iv) $\langle f \mid g\mathbf{q}\rangle = \langle f \mid g\rangle\mathbf{q}$
(v) $\langle f\mathbf{q} \mid g\rangle = \overline{\mathbf{q}}\langle f \mid g\rangle$

where $\overline{\mathbf{q}}$ stands for the quaternionic conjugate. We assume that the space V_H^R is complete under the norm given above. Then, together with $\langle \cdot \mid \cdot\rangle$ this defines a right quaternionic Hilbert space, which we shall assume to be separable. Quaternionic Hilbert spaces share most of the standard properties of complex Hilbert spaces. In particular, the Cauchy-Schwartz inequality holds on quaternionic Hilbert spaces as well as the Riesz representation theorem for their duals. Thus, the Dirac bra-ket notation can be adapted to quaternionic Hilbert spaces:

$$\mid f\mathbf{q}\rangle = \mid f\rangle\mathbf{q}, \qquad \langle f\mathbf{q} \mid = \overline{\mathbf{q}}\langle f \mid\,,$$

for a right quaternionic Hilbert space, with $|f\rangle$ denoting the vector f and $\langle f|$ its dual vector. Let O_R be an operator on a right quaternionic Hilbert space. The scalar multiplication of O_R should be written as $\mathbf{q}O_R$ and the action must take the form [6]

$$(\mathbf{q}O_R) \mid f\rangle = (O_R \mid f\rangle)\overline{\mathbf{q}}. \tag{2.9}$$

The adjoint $O_R^\dagger$ of O_R is defined as

$$\langle g \mid O_R f\rangle = \langle O_R^\dagger g \mid f\rangle; \quad \text{for all } f, g \in V_H^R.$$

2.2.2. *Left Quaternionic Hilbert Space.* Let V_H^L be a linear vector space under left multiplication by quaternionic scalars. For $f, g, h \in V_H^L$ and $\mathbf{q} \in H$, the inner product

$$\langle \cdot \mid \cdot \rangle : V_H^L \times V_H^L \longrightarrow H$$

satisfies the following properties

(i) $\overline{\langle f \mid g\rangle} = \langle g \mid f\rangle$
(ii) $\|f\|^2 = \langle f \mid f\rangle > 0$ unless $f = 0$, a real norm
(iii) $\langle f \mid g + h\rangle = \langle f \mid g\rangle + \langle f \mid h\rangle$
(iv) $\langle \mathbf{q}f \mid g\rangle = \mathbf{q}\langle f \mid g\rangle$
(v) $\langle f \mid \mathbf{q}g\rangle = \langle f \mid g\rangle\overline{\mathbf{q}}$

Again, we shall assume that the space V_H^L together with $\langle \cdot \mid \cdot\rangle$ is a separable Hilbert space. Also,

$$\mid \mathbf{q}f\rangle = \mid f\rangle\overline{\mathbf{q}}, \qquad \langle \mathbf{q}f \mid = \mathbf{q}\langle f \mid\,. \tag{2.10}$$

Note that, because of our convention for inner products, for a left quaternionic Hilbert space, the bra vector $\langle f \mid$ is to be identified with the vector itself, while the ket vector $\mid f\rangle$ is to be identified with its dual. Note also that there is a natural left multiplication by quaternionic scalars on the dual of a right quaternionic Hilbert space and a similar right multiplication on the dual of a left quaternionic Hilbert space.

Separable quaternionic Hilbert spaces admit countable orthonormal bases. Let V_H^R be a right quaternionic Hilbert space and let $\{e_v\}_{\nu=0}^N$ (N could be finite or infinite) be an orthonormal basis for it. Then, $\langle e_\nu \mid e_\mu\rangle = \delta_{\nu\mu}$ and any vector $f \in V_H^R$ has the expansion $f = \sum_\nu e_\nu f_\nu$, with $f_\nu = \langle e_\nu \mid f\rangle \in H$. Using such a basis, it is possible to introduce a multiplication from the left on V_H^R by elements of H. Indeed, for $f \in V_H^R$ and $\mathbf{q} \in H$ we define,

$$\mathbf{q}f = \sum_{\nu} e_\nu(\mathbf{q}f_\nu). \tag{2.11}$$

Further, $\langle \mathbf{q}f \mid g\rangle = \langle f \mid \overline{\mathbf{q}}g\rangle$ (see [18]). The field of quaternions H itself can be turned into a left quaternionic Hilbert space by defining the inner product $\langle \mathbf{q} \mid \mathbf{q}'\rangle = \mathbf{q}\mathbf{q}'^\dagger = \mathbf{q}\overline{\mathbf{q}'}$ or into a right quaternionic Hilbert space with $\langle \mathbf{q} \mid \mathbf{q}'\rangle = \mathbf{q}^\dagger\mathbf{q}' = \overline{\mathbf{q}}\mathbf{q}'$.

2.2.3. *Quaternionic Hilbert Spaces of Square Integrable Functions.* Let (X, μ) be a measure space and H the field of quaternions, then

$$\left\{ f : X \to H \,\middle|\, \int_X |f(x)|^2 d\mu(x) < \infty \right\}$$

is a right quaternionic Hilbert space which is denoted by $L_H^2(X, \mu)$, with the (right) scalar product

$$\langle f \mid g\rangle = \int_X \overline{f(x)}g(x)d\mu(x), \tag{2.12}$$

where $\overline{f(x)}$ is the quaternionic conjugate of $f(x)$, and (right) scalar multiplication fa, $a \in H$, with $(fa)(q) = f(q)a$ (see [14, 17] for details). Similarly, one could define a left quaternionic Hilbert space of square integrable functions.

3. Coherent states on right quaternion Hilbert spaces

The main content of this section is extracted from [20] as needed here. For an enhanced explanation we refer the reader to [20]. In [20] the authors have defined coherent states on V_H^R and V_H^L, and also established the normalization and resolution of the identities for each of them. We briefly revisit the coherent states of V_H^R and the normalization and resolution of the identity. Let $\{\mid f_m\rangle\}_{m=0}^\infty$ be an orthonormal basis of V_H^R. For $\mathbf{q} \in V_H^R$, the coherent states are defined as vectors in V_H^R in the form of

$$\mid \mathbf{q}\rangle = \mathcal{N}(\mid \mathbf{q} \mid)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \mid f_m\rangle \frac{\mathbf{q}^m}{\sqrt{\rho(m)}}, \tag{3.1}$$

where $\mathcal{N}(|\mathbf{q}|)$ is the normalization factor and $\{\rho(m)\}_{m=0}^\infty$ is a positive sequence of real numbers. Using conditions (2.6), we can determine the normalization factor $\mathcal{N}(\mid \mathbf{q} \mid)$, and the resolution of the identity. In order for the norm of $\mid \mathbf{q}\rangle$ to be finite, we must have

$$\langle \mathbf{q} \mid \mathbf{q}\rangle = \mathcal{N}(\mid \mathbf{q} \mid)^{-1} \sum_{m=0}^{\infty} \frac{r^{2m}}{\rho(m)} < \infty. \tag{3.2}$$

Therefore, if the positive sequence $\{\rho(m)\}_{m=0}^\infty$ of real numbers converges to $\ell > 0$, then we are required to restrict the domain into

$$\mathcal{D} = [0, \sqrt{\ell}\,) \times [0, \pi] \times [0, 2\pi)^2 \tag{3.3}$$

so that the convergence of the above series is guaranteed. The typical measure (2.7) is an appropriate one on the domain $\mathcal{D}$ too. By requiring $\langle \mathbf{q} \mid \mathbf{q}\rangle = 1$, the normalization factor is obtained as

$$\mathcal{N}(\mid \mathbf{q} \mid) = \sum_{m=0}^{\infty} \frac{r^{2m}}{\rho(m)}. \tag{3.4}$$

Using the measure $d\varsigma(r, \theta, \phi, \psi)$ one can obtain the following operator valued integral on the domain $\mathcal{D}$ of (3.3):

$$\int_{\mathcal{D}} \mid \mathbf{q}\rangle\langle \mathbf{q} \mid d\varsigma(r, \theta, \phi, \psi) = \sum_{m=0}^{\infty} \frac{2\pi}{\rho(m)} I_m, \tag{3.5}$$

where I_m is

$$\int_0^{\sqrt{\ell}} \frac{r^{2m}}{\mathcal{N}(\mid \mathbf{q} \mid)} \mid f_m\rangle\langle f_m \mid d\tau(r),$$

and in obtaining it we have used the identity

$$\iiint_E e^{i(m-l)\theta\sigma(\hat{n})} \sin\phi \, d\phi \, d\theta \, d\varphi = 2\pi\delta_{ml}\mathbb{I}_2, \tag{3.6}$$

where δ_{ml} is the Kronecker's delta and $E = [0, 2\pi) \times [0, \pi) \times [0, 2\pi)$. The resolution of the identity,

$$\int_{\mathcal{D}} \mid \mathbf{q}\rangle\langle \mathbf{q} \mid d\varsigma(r, \theta, \phi, \psi) = \mathbb{I}_{V_D^R}, \tag{3.7}$$

where $\mathbb{I}_{V_D^R}$ is the identity operator on V_D^R, is obtained if there is a measure to satisfy the moment problem,

$$\frac{2\pi}{\rho(m)} \int_0^{\sqrt{\ell}} \frac{r^{2m}}{\mathcal{N}(\mid \mathbf{q} \mid)} d\tau(r)\mathbb{I}_2 = \mathbb{I}_2. \tag{3.8}$$

If the measure $d\tau(r)$ is chosen such that

$$d\tau(r) = \frac{\mathcal{N}(\mid \mathbf{q} \mid)}{2\pi} \lambda(r)dr, \tag{3.9}$$

then there exists an auxiliary density $\lambda(r)$ to solve (3.8), that is, we get

$$\int_0^{\sqrt{\ell}} r^{2m}\lambda(r)dr\mathbb{I}_2 = \rho(m)\mathbb{I}_2. \tag{3.10}$$

Particularly, if $\rho(m) = m!$, then the normalization factor $\mathcal{N}(\mid \mathbf{q} \mid) = e^{|\mathbf{q}|^2}$ and $\ell = \infty$. The resolution of the identity can be established for (3.1) with $\lambda(r) = 2re^{-r^2}$. In this case $\mathcal{D} = H$ and the CS are called *right quaternionic canonical coherent states (RQCS)*. For the purpose of quatizing the quaternions we shall use these canonical set of CS.

4. Coherent state quantization: General scheme

Let (X, μ) be a measure space and $L^2(X, \mu)$ be given by

$$\left\{ f : X \to \mathbb{C} \mid \int_X |f(x)|^2 d\mu(x) < \infty \right\}.$$

The Berezin-Toeplitz or anti-Wick or coherent state quantization, as used by various authors in the literature, associates a classical observable that is a function $f(x)$ on X to an operator valued integral. We continue with the general procedure described in [3] and applied, for example, in [9, 8, 7].

Choose a countable orthonormal basis

$$\mathcal{O} = \{\phi_n \mid n = 0, 1, 2 \cdots\}$$

in $L^2(X, \mu)$, that is

$$\langle\phi_n|\phi_m\rangle = \int_X \overline{\phi_n(x)}\phi_m(x)d\mu(x) = \delta_{mn}, \tag{4.1}$$

and assume that

$$0 < \sum_{n=0}^{\infty} |\phi_n(x)|^2 := \mathcal{N}(x) < \infty \quad a.e. \tag{4.2}$$

holds. Let $\mathfrak{H}$ be a separable complex Hilbert space with orthonormal basis $\{|e_n\rangle \mid n = 0, 1, 2 \cdots\}$ in 1-1 correspondence with $\mathcal{O}$. In particular $\mathfrak{H}$ can be taken as $\mathfrak{H} = \overline{\text{span}\mathcal{O}}$ in $L^2(X, \mu)$, where the bar stands for the closure. Then the family $\mathcal{F}_{\mathfrak{H}} = \{|x\rangle \mid x \in X\}$ with

$$|x\rangle = \mathcal{N}(x)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \overline{\phi_n(x)}|e_n\rangle \in \mathfrak{H} \tag{4.3}$$

forms a set of coherent states(CS). From (4.1) and (4.2) we have

$$\langle x|x\rangle = 1 \tag{4.4}$$

$$\int_X \mathcal{N}(x)|x\rangle\langle x|d\mu(x) = \mathbb{I}_{\mathfrak{H}}, \tag{4.5}$$

where $\mathbb{I}_{\mathfrak{H}}$ is the identity operator on $\mathfrak{H}$. We call the set $\mathcal{F}_{\mathfrak{H}}$ a set of CS only for satisfying the normalization and a resolution of the identy. Equation (4.5) allows us to implement CS or frame quantization of the set of parameters X by associating a function

$$X \ni x \mapsto f(x)$$

that satisfies appropriate conditions the following operator in $\mathfrak{H}$

$$f(x) \mapsto A_f = \int_X \mathcal{N}(x)f(x)|x\rangle\langle x|d\mu(x). \tag{4.6}$$

The matrix elements of A_f with respect to the basis $\{|e_n\rangle\}$ are give by

$$\begin{aligned}(A_f)_{mn} &= \langle e_m|A_f|e_n\rangle \\ &= \int_X f(x)\overline{\phi_m(x)}\phi_n(x)d\mu(x).\end{aligned}$$

The operator A_f is

(a) symmetric if $f(x)$ is real valued;
(b) bounded if $f(x)$ is bounded;
(c) self-adjoint if $f(x)$ is real semi-bounded (through Friedrich's extension).

In order to view the upper symbol f of A_f as a quantizable object (with respect to the family $\mathcal{F}_{\mathfrak{H}}$) a reasonable requirement is that the so-called lower symbol of A_f defined as

$$\begin{aligned}\check{f}(x) &= \langle x|A_f|x\rangle \\ &= \int_X \mathcal{N}(x')f(x')|\langle x|x'\rangle|^2 d\mu(x')\end{aligned}$$

be a smooth function on X with respect to some topology assigned to the set X. Associating to the classical observable $f(x)$ the mean value $\langle x|A_f|x\rangle$ one can also get the so-called Berezin transform $B[f]$ with $B[f](x) = \langle x|A_f|x\rangle$, for example, see [15] for details.

5. Quantization of the quaternions

In this section we shall adapt the general procedure outlined in the above setion to quaternions. Since $(H, d\varsigma(r, \theta, \phi, \psi))$ is a measure space, the set

$$\left\{ f : H \to H \mid \int_H |f(\mathbf{q})|^2 d\varsigma(r, \theta, \phi, \psi) < \infty \right\}$$

is the space of right quaternionic square integrable functions and is denoted by $L^2_H(H, d\varsigma(r, \theta, \phi, \psi))$. Define the sequence of functions $\{\phi_n\}_{n=0}^{\infty}$ such that

$$\phi_n : H \longrightarrow H$$

by

$$\phi_n(\mathbf{q}) = \frac{\overline{\mathbf{q}}^n}{\sqrt{n!}}, \quad \text{for all } \mathbf{q} \in H. \tag{5.1}$$

Then $\phi_n \in L^2_H(H, d\varsigma(r, \theta, \phi, \psi))$, for all $n = 0, 1, 2 \cdots$ and from (3.6) $\langle\phi_m \mid \phi_n\rangle = \delta_{mn}$ (see [20]). That is,

$$\mathcal{O} = \{\phi_n \mid n = 0, 1, 2 \cdots\}$$

is an orthonormal set in $L^2_H(H, d\varsigma(r,\theta,\phi,\psi))$. The right quaternionic span of $\mathcal{O}$ is the space of anti-right-regular functions [19] (the counter part of complex anti-holomorphic functions). Let $\mathfrak{H}$ be a separable right quaternionic Hilbert space with an orthonormal basis

$$\mathcal{E} = \{\mid e_n\rangle \mid n = 0, 1, 2\cdots\}$$

which is in $1-1$ correspondence with $\mathcal{O}$. Then the coherent states (3.1) become

$$(5.2)\qquad \mid\gamma_{\mathbf{q}}\rangle = e^{|\mathbf{q}|^{-\frac{1}{2}}}\sum_{m=0}^{\infty}\mid e_m\rangle\overline{\phi_m}.$$

Using the set of CS (5.2) we shall establish the coherent state quantization on $\mathfrak{H}$ by associating a function

$$H \ni \mathbf{q} \longmapsto f(\mathbf{q},\overline{\mathbf{q}}).$$

Now let us define the operator on $\mathfrak{H}$ by

$$(5.3)\qquad f(\mathbf{q},\overline{\mathbf{q}}) \mapsto A_f,$$

where A_f is given by the operator valued integral

$$(5.4)\quad A_f = \int_H \mid\gamma_{\mathbf{q}}\rangle f(\mathbf{q},\overline{\mathbf{q}})\langle\gamma_{\mathbf{q}}\mid d\varsigma(r,\theta,\phi,\psi).$$

Remark 5.1. The operator A_f is formed by the vector $\mid\gamma_{\mathbf{q}}\rangle f(\mathbf{q},\overline{\mathbf{q}})$, which is the right scalar multiple of the vector $\mid\gamma_{\mathbf{q}}\rangle$ by the scalar $f(\mathbf{q},\overline{\mathbf{q}})$, and the dual vector $\langle\gamma_{\mathbf{q}}\mid$. Instead if one takes

$$(5.5)\quad A_f = \int_H f(\mathbf{q},\overline{\mathbf{q}})\mid\gamma_{\mathbf{q}}\rangle\langle\gamma_{\mathbf{q}}\mid d\varsigma(r,\theta,\phi,\psi),$$

then it is formed by $f(\mathbf{q},\overline{\mathbf{q}})\mid\gamma_{\mathbf{q}}\rangle$ (a left scalar multiple of a right Hilbert space vector) and the dual vector $\langle\gamma_{\mathbf{q}}\mid$, which is unconventional . Further, due to the noncommutativity of quaternions, the A_f in the form (5.5) shall cause severe technical problems in the follow up computations.

Now

$$\begin{aligned} A_f &= \int_H \mid\gamma_{\mathbf{q}}\rangle f(\mathbf{q},\overline{\mathbf{q}})\langle\gamma_{\mathbf{q}}\mid d\varsigma(r,\theta,\phi,\psi) \\ &= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{\mid e_m\rangle J_{m,l}\langle e_l\mid}{\sqrt{m!\;l!}}; \end{aligned}$$

where the integral $J_{m,l}$ is given by

$$\iiiint\limits_{[0,\infty)\times[0,\pi]\times[0,2\pi)^2}\frac{\mathbf{q}^m f(\mathbf{q},\overline{\mathbf{q}})\overline{\mathbf{q}}^l}{e^{r^2}}d\varsigma(r,\theta,\phi,\psi).$$

By direct calculation we have that if $f(\mathbf{q},\overline{\mathbf{q}}) = \mathbf{q}$, then

$$(5.6)\qquad A_{\mathbf{q}} = \sum_{m=0}^{\infty}\sqrt{(m+1)}\mid e_m\rangle\langle e_{m+1}\mid$$

and if $f(\mathbf{q},\overline{\mathbf{q}}) = \overline{\mathbf{q}}$, then

$$(5.7)\qquad A_{\overline{\mathbf{q}}} = \sum_{m=0}^{\infty}\sqrt{(m+1)}\mid e_{m+1}\rangle\langle e_m\mid.$$

Moreover if $f(\mathbf{q},\overline{\mathbf{q}}) = 1$, then $A_1 = \mathbb{I}_{\mathfrak{H}}$. It should be mentioned that, since the operator A_f is a quaternionic operator, the usual properties of its complex counterpart may not hold. In this regard, each property used must be validated. First let $|f\rangle, |g\rangle \in \mathfrak{H}$. Since $\mathfrak{H}$ is a right Hilbert space, there are scalars $\{\alpha_l\}, \{\beta_j\}$ in H such that

$$|f\rangle = \sum_{l=0}^{\infty}|e_l\rangle\alpha_l \quad\text{and}\quad |g\rangle = \sum_{j=0}^{\infty}|e_j\rangle\beta_j.$$

With these it can be seen that

$$\begin{aligned} \langle A_{\overline{\mathbf{q}}}g\mid f\rangle &= \langle g\mid A_{\mathbf{q}}f\rangle \\ &= \sum_{m=0}^{\infty}\overline{\beta}_l\alpha_{m+1}\sqrt{m+1}. \end{aligned}$$

That is,

$$\langle A_{\overline{\mathbf{q}}}g\mid f\rangle = \langle g\mid A_{\mathbf{q}}f\rangle;\quad \text{for all}\quad |f\rangle, |g\rangle \in \mathfrak{H}.$$

Hence $A_{\overline{\mathbf{q}}}$ is the adjoint of $A_{\mathbf{q}}$ and vice-versa. Now A_f is an operator from $\mathfrak{H}$ to $\mathfrak{H}$, and if $\mathfrak{H} = \overline{span\mathcal{O}}$ (right linear span over H), then it is a subspace of $L^2_H(H, d\varsigma(r,\theta,\phi,\psi))$. That is,

$$A_f : \mathfrak{H} \longrightarrow \mathfrak{H} \;\text{ by }\; A_f(u) = A_f\mid u\rangle,$$

for all $|u\rangle \in \mathfrak{H}$. Hence, $A_f(u)$ will be determined by this integral

$$\int_H \mid\gamma_{\mathbf{q}}\rangle f(\mathbf{q},\overline{\mathbf{q}})\langle\gamma_{\mathbf{q}}\mid u\rangle d\varsigma(r,\theta,\phi,\psi).$$

Moreover, for each $|u\rangle \in \mathfrak{H}$, $A_f\mid u\rangle \in \mathfrak{H}$. For $|u\rangle, |v\rangle \in \mathfrak{H}$, it can also be considered as a function

$$A_f : \mathfrak{H}\times\mathfrak{H} \longrightarrow H \;\text{ by }\; A_f(u,v) = \langle u\mid A_f\mid v\rangle.$$

Thereby, $A_f(u,v)$ will be determined by the quaternion valued integral

$$\int_H \langle u\mid\gamma_{\mathbf{q}}\rangle f(\mathbf{q},\overline{\mathbf{q}})\langle\gamma_{\mathbf{q}}\mid v\rangle d\varsigma(r,\theta,\phi,\psi).$$

Since $\mid\gamma_{\mathbf{q}}\rangle$ is a column vector and $\langle\gamma_{\mathbf{q}}\mid$ is a row vector, we can see that the operator A_f is a matrix and the matrix elements with respect to the basis $\{\mid e_n\rangle\}$ are given by

$$(A_f)_{mn} = \langle e_m\mid A_f\mid e_n\rangle.$$

That is, $(A_f)_{mn}$ is determined by the integral

$$\int_H \langle e_m \mid \gamma_{\mathbf{q}}\rangle f(\mathbf{q},\overline{\mathbf{q}})\langle \gamma_{\mathbf{q}} \mid e_n\rangle d\varsigma(r,\theta,\phi,\psi).$$

We have

$$\langle e_m \mid \gamma_{\mathbf{q}}\rangle = \mathcal{N}(\mid \mathbf{q} \mid)^{-\frac{1}{2}}\ \overline{\phi_m(\mathbf{q})}$$

and

$$\langle \gamma_{\mathbf{q}} \mid e_n\rangle = \overline{\langle e_n \mid \gamma_{\mathbf{q}}\rangle} = \mathcal{N}(\mid \mathbf{q} \mid)^{-\frac{1}{2}}\ \phi_n(\mathbf{q}).$$

Therefore, $(A_f)_{mn}$ is given by

$$\int_H \mathcal{N}(\mid \mathbf{q} \mid)^{-1}\overline{\phi_m(\mathbf{q})} f(\mathbf{q},\overline{\mathbf{q}})\phi_n(\mathbf{q}).d\varsigma(r,\theta,\phi,\psi).$$

Hence, it can easily be seen that

$$\begin{aligned}(A_{\mathbf{q}})_{k,l} &= \begin{cases} \sqrt{k+1} & \text{if} \quad l=k+1 \\ 0 & \text{if} \quad l\neq k+1, \end{cases} \\ (A_{\overline{\mathbf{q}}})_{k,l} &= \begin{cases} \sqrt{k} & \text{if} \quad l=k-1 \\ 0 & \text{if} \quad l\neq k-1. \end{cases}\end{aligned}$$

Let us realize the operator A_f as annihilation and creation operators. From (5.6) and (5.7) we have $A_{\mathbf{q}} \mid e_0\rangle = 0$,

$$A_{\mathbf{q}} \mid e_m\rangle = \sqrt{m} \mid e_{m-1}\rangle\ ;\ m=1,2,\cdots$$

and

$$A_{\overline{\mathbf{q}}} \mid e_m\rangle = \sqrt{m+1} \mid e_{m+1}\rangle\ ;\ m=0,1,2,\cdots$$

That is, $A_{\mathbf{q}}, A_{\overline{\mathbf{q}}}$ are annihilation and creation operators respectively. Moreover, one can easily see that $A_{\mathbf{q}} \mid \gamma_{\mathbf{q}}\rangle = \mid \gamma_{\mathbf{q}}\rangle\mathbf{q}$, which is in complete analogy with the action of the annihilation operator on the ordinary harmonic oscillator CS and the result obtained in [20]. Now a direct calculation shows that

$$A_{\mathbf{q}}A_{\overline{\mathbf{q}}} = \sum_{m=0}^{\infty}(m+1) \mid e_m\rangle\langle e_m \mid$$

and

$$A_{\overline{\mathbf{q}}}A_{\mathbf{q}} = \sum_{m=0}^{\infty}(m+1) \mid e_{m+1}\rangle\langle e_{m+1} \mid .$$

Thereby the commutator of $A_{\overline{\mathbf{q}}}, A_{\mathbf{q}}$ takes the form

$$\begin{aligned}[A_{\mathbf{q}}, A_{\overline{\mathbf{q}}}] &= A_{\mathbf{q}}A_{\overline{\mathbf{q}}} - A_{\overline{\mathbf{q}}}A_{\mathbf{q}} \\ &= \sum_{m=0}^{\infty} \mid e_m\rangle\langle e_m \mid = \mathbb{I}_{\mathfrak{H}}.\end{aligned}$$

5.1. **Number, position and momentum operators and Hamiltonian.** Let $N = A_{\overline{\mathbf{q}}}A_{\mathbf{q}}$, then we have

$$\begin{aligned} N \mid e_k\rangle &= A_{\overline{\mathbf{q}}}A_{\mathbf{q}} \mid e_k\rangle \\ &= \sum_{m=0}^{\infty} \mid e_{m+1}\rangle\langle e_{m+1} \mid e_k\rangle(m+1) \\ &= \mid e_k\rangle k. \end{aligned}$$

Thereby N acts as the number operator and the Hilbert space $\mathfrak{H}$ is the quaternionic Fock space (for quaternion Fock spaces see [21]). As an analogue of the usual harmonic oscillator Hamiltonian, if we take $\hat{H} = N + \mathbb{I}_{\mathfrak{H}}$, then $\hat{H} \mid e_n\rangle = \mid e_n\rangle(n+1)$, which is a Hamiltonian in the right quaternionic Hilbert space $\mathfrak{H}$ with spectrum $(n+1)$ and eigenvector $\mid e_n\rangle$.

Remark 5.2. In the complex quantum mechanics, for the canonical CS, $\mid z\rangle$, $z\in\mathbb{C}$, the lower symbol or the expectation value of the number operator, $\langle z|N|z\rangle$, is precisely $|z|^2$. The position and momentum coordinates are $q = \frac{1}{\sqrt{2}}(z+\overline{z})$ and $p = \frac{-i}{\sqrt{2}}(z-\overline{z})$ and by linearity one infers that the position and momentum operators as $Q = \frac{1}{\sqrt{2}}(A_z+A_{\overline{z}})$ and $P = \frac{-i}{\sqrt{2}}(A_z-A_{\overline{z}})$. The CS quantized classical harmonic oscillator, $\hat{H} = \frac{1}{2}(q^2+p^2)$ is $A_{\hat{H}} = A_{|z|^2} = N + \mathbb{I}_{\mathfrak{H}}$, where $I_{\mathfrak{H}}$ is the identity operator of the Fock space. The operators Q and P satisfy the commutation rule $[Q,P] = i\mathbb{I}_{\mathfrak{H}}$ and are self-adjoint. If one simply takes the canonical quantization of the classical Hamiltonian it becomes $\hat{H} = \frac{1}{2}(Q^2+P^2) = N + \frac{1}{2}\mathbb{I}_{\mathfrak{H}}$. For details we refer the reader to [9, 3].

In the case of quaternions we have three imaginary units, i, j and k, and if one try to duplicate the position and momentum coordinates with one of i, j or k, that is, if we take

$$\mathfrak{q} = \frac{1}{\sqrt{2}}(\mathbf{q}+\overline{\mathbf{q}}) \ \text{ and } \ \mathfrak{p} = \frac{-i}{\sqrt{2}}(\mathbf{q}-\overline{\mathbf{q}}),$$

then a simple calculation shows that $\hat{H} = \frac{1}{2}(\mathfrak{q}^2+\mathfrak{p}^2) \neq |\mathbf{q}|^2$. However, the lower symbol of N is $\langle \gamma_{\mathbf{q}} \mid N \mid \gamma_{\mathbf{q}}\rangle = |\mathbf{q}|^2$ and through a rather lengthy calculation we can see that $A_{|\mathbf{q}|^2} = N + \mathbb{I}_{\mathfrak{H}}$. The best way to avoid the difficulty in defining the position and momentum coordinates is to consider quaternion slices.

Remark 5.3. From now on we restrict the analysis to a quaternion slice L_I. However, we shall be using the same symbols for notational convenience. The reader should understand it in the following sense:

$$|\gamma_{\mathbf{q}}\rangle = |\gamma_{\mathbf{q}}\rangle\,|_{L_I}\;,$$

which still forms a set of RQCS with the same normalization factor with $\mathbf{q}\in L_I$ and a resolution of identity with the measure $d\mu(r,\theta)=\frac{1}{2\pi}re^{-r^2}drd\theta$ and the identity operator $\mathbb{I}_{\mathfrak{H}}^{I}$, which is the identity operator on the Hilbert space $\mathfrak{H}$ over the field L_I. All other operators should also be understood in the same way. That is

$$A_{\mathbf{q}}=A_{\mathbf{q}}^{I},\;A_{\bar{\mathbf{q}}}=A_{\bar{\mathbf{q}}}^{I},\;,Q=Q_I,\;P=P_I.$$

In this regard, let $\mathbf{q}\in H$, then there exists $I\in\mathbb{S}$ such that $\mathbf{q}=x+Iy$ for some $x,y\in\mathbb{R}$. Now note that $\mathbf{q}I=(x+Iy)I=I(x+Iy)=I\mathbf{q}$, similarly $\overline{\mathbf{q}}I=(x-Iy)I=I(x-Iy)=I\overline{\mathbf{q}}$. That is, the commutativity holds among I, $\mathbf{q}$ and $\overline{\mathbf{q}}$. Let us define the position and momentum coordinates by

$$\mathfrak{q}=\frac{1}{\sqrt{2}}(\mathbf{q}+\overline{\mathbf{q}})\quad\text{and}\quad\mathfrak{p}=\frac{-I}{\sqrt{2}}(\mathbf{q}-\overline{\mathbf{q}}),$$

then, with the aid of the commutativity among $I,\mathbf{q}$ and $\overline{\mathbf{q}}$, the Hamiltonian can be calculated as

$$\hat{H}=\frac{1}{2}\left(\mathfrak{q}^2+\mathfrak{p}^2\right)=|\mathbf{q}|^2.$$

Recall that on a right quaternionic Hilbert space operators are multiplied on the left by quaternion scalars. From the position and momentum coordinates, using linearity, we get the position operator, Q, and the momentum operator, P, as

$$\begin{aligned}Q&=\frac{1}{\sqrt{2}}\left(A_{\mathbf{q}}+A_{\overline{\mathbf{q}}}\right)\quad\text{and}\\ P&=\frac{-I}{\sqrt{2}}\left(A_{\mathbf{q}}-A_{\overline{\mathbf{q}}}\right).\end{aligned}$$

Since $(A_{\overline{\mathbf{q}}})^{\dagger}=A_{\mathbf{q}}$ and $(-I)^{\dagger}=I$, the operators P and Q are self-adjoint. Using the fact (2.9) we can see that $A_{\overline{\mathbf{q}}}(IA_{\mathbf{q}})=IA_{\overline{\mathbf{q}}}A_{\mathbf{q}}$. With the aid of this we get

$$\begin{aligned}QP&=\left[\frac{(A_{\mathbf{q}}+A_{\overline{\mathbf{q}}})}{\sqrt{2}}\right]\left[-I\frac{(A_{\mathbf{q}}-A_{\overline{\mathbf{q}}})}{\sqrt{2}}\right]\\ &=-\frac{1}{2}I\left[{A_{\mathbf{q}}}^2+A_{\overline{\mathbf{q}}}A_{\mathbf{q}}-A_{\mathbf{q}}A_{\overline{\mathbf{q}}}-{A_{\overline{\mathbf{q}}}}^2\right]\end{aligned}$$

and

$$\begin{aligned}PQ&=\left[-I\frac{(A_{\mathbf{q}}-A_{\overline{\mathbf{q}}})}{\sqrt{2}}\right]\left[\frac{(A_{\mathbf{q}}+A_{\overline{\mathbf{q}}})}{\sqrt{2}}\right]\\ &=-\frac{1}{2}I\left[{A_{\mathbf{q}}}^2-A_{\overline{\mathbf{q}}}A_{\mathbf{q}}+A_{\mathbf{q}}A_{\overline{\mathbf{q}}}-{A_{\overline{\mathbf{q}}}}^2\right].\end{aligned}$$

Thereby we have the commutator

$$[Q,P]=QP-PQ=I\left[A_{\mathbf{q}},A_{\overline{\mathbf{q}}}\right]=I\mathbb{I}_{\mathfrak{H}}.$$

We also have

$$\begin{aligned}Q^2&=\frac{1}{2}\left[{A_{\mathbf{q}}}^2+A_{\overline{\mathbf{q}}}A_{\mathbf{q}}+A_{\mathbf{q}}A_{\overline{\mathbf{q}}}+{A_{\overline{\mathbf{q}}}}^2\right]\quad\text{and}\\ P^2&=-\frac{1}{2}\left[{A_{\mathbf{q}}}^2-A_{\overline{\mathbf{q}}}A_{\mathbf{q}}-A_{\mathbf{q}}A_{\overline{\mathbf{q}}}+{A_{\overline{\mathbf{q}}}}^2\right]\end{aligned}$$

Hence

$$\begin{aligned}\hat{H}&=\frac{Q^2+P^2}{2}=\frac{1}{2}[A_{\overline{\mathbf{q}}}A_{\mathbf{q}}+A_{\mathbf{q}}A_{\overline{\mathbf{q}}}]\\ &=A_{\overline{\mathbf{q}}}A_{\mathbf{q}}+\frac{1}{2}[A_{\mathbf{q}}A_{\overline{\mathbf{q}}}-A_{\overline{\mathbf{q}}}A_{\mathbf{q}}]\\ &=N+\frac{1}{2}\mathbb{I}_{\mathfrak{H}},\end{aligned}$$

which is in complete analogy with the complex case in the sense of *canonical quantization*, which simply replaces the classical coordinates by quantum observables (corresponding self-adjoint operators).

5.2. **Heisenberg uncertainty.** In the following we shall show that the RQCS saturate the Heisenberg uncertainty relation and thereby they form a set of intelligent states. We shall also demonstrate that the RQCS are, in fact, minimum uncertainty states. In order to compute the expectation values of the involved operators recall that $A_{\mathbf{q}}|e_0\rangle=0$,

$$\begin{aligned}A_{\mathbf{q}}|e_m\rangle&=\sqrt{m}|e_{m-1}\rangle;\quad m=1,2,\cdots\\ A_{\bar{\mathbf{q}}}|e_m\rangle&=\sqrt{m+1}|e_{m+1}\rangle;\quad m=0,1,\cdots\end{aligned}$$

and

$$A_{\mathbf{q}}|\gamma_{\mathbf{q}}\rangle=|\gamma_{\mathbf{q}}\rangle\mathbf{q}. \tag{5.8}$$

Using (5.8) we can easily see that

$$A_{\mathbf{q}}^2|\gamma_{\mathbf{q}}\rangle=A_{\mathbf{q}}|\gamma_{\mathbf{q}}\rangle\mathbf{q}=|\gamma_{\mathbf{q}}\rangle\mathbf{q}^2.$$

Hence, as $\langle\gamma_{\mathbf{q}}|\gamma_{\mathbf{q}}\rangle=1$, we get

$$\langle\gamma_{\mathbf{q}}|A_{\mathbf{q}}|\gamma_{\mathbf{q}}\rangle=\mathbf{q}\quad\text{and}\quad\langle\gamma_{\mathbf{q}}|A_{\mathbf{q}}^2|\gamma_{\mathbf{q}}\rangle=\mathbf{q}^2.$$

For the purpose of fitting long expressions in double column, we let $a_m=\sqrt{m+1}$ and $b_m=\sqrt{(m+1)(m+2)}$. The action of the operators, $A_{\bar{\mathbf{q}}},A_{\bar{\mathbf{q}}}^2,A_{\bar{\mathbf{q}}}A_{\mathbf{q}}$ and $A_{\mathbf{q}}A_{\bar{\mathbf{q}}}$ on the RQCS takes the form

$$\begin{aligned}A_{\bar{\mathbf{q}}}|\gamma_{\mathbf{q}}\rangle&=e^{-|\mathbf{q}|^2/2}\sum_{m=0}^{\infty}A_{\bar{\mathbf{q}}}|e_m\rangle\frac{\mathbf{q}^m}{\sqrt{m!}}\\ &=e^{-|\mathbf{q}|^2/2}\sum_{m=0}^{\infty}|e_{m+1}\rangle a_m\frac{\mathbf{q}^m}{\sqrt{m!}},\end{aligned}$$

and similarly,

$$A^2_{\bar{\mathbf{q}}}|\gamma_{\mathbf{q}}\rangle = e^{-|\mathbf{q}|^2/2}\sum_{m=0}^{\infty}|e_{m+2}\rangle b_m\frac{\mathbf{q}^m}{\sqrt{m!}},$$

$$A_{\bar{\mathbf{q}}}A_{\mathbf{q}}|\gamma_{\mathbf{q}}\rangle = e^{-|\mathbf{q}|^2/2}\sum_{m=0}^{\infty}|e_{m+1}\rangle a_m\frac{\mathbf{q}^{m+1}}{\sqrt{m!}}$$

and

$$A_{\mathbf{q}}A_{\bar{\mathbf{q}}}|\gamma_{\mathbf{q}}\rangle = e^{-|\mathbf{q}|^2/2}\sum_{m=0}^{\infty}|e_m\rangle a_m^2\frac{\mathbf{q}^m}{\sqrt{m!}}.$$

The dual of the CS is

$$\langle\gamma_{\mathbf{q}}| = e^{-|\mathbf{q}|^2/2}\sum_{m=0}^{\infty}\frac{\bar{\mathbf{q}}^m}{\sqrt{m!}}\langle e_m|.$$

Thereby we get the expectation values

$$\begin{aligned}
&\langle\gamma_{\mathbf{q}}|A_{\bar{\mathbf{q}}}|\gamma_{\mathbf{q}}\rangle \\
&= e^{-|\mathbf{q}|^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\bar{\mathbf{q}}^m}{\sqrt{m!}}\langle e_m|e_{n+1}\rangle a_n\frac{\mathbf{q}^n}{\sqrt{n!}} \\
&= e^{-|\mathbf{q}|^2}\sum_{m=0}^{\infty}\frac{\bar{\mathbf{q}}^{m+1}\mathbf{q}^m}{m!} \\
&= e^{-|\mathbf{q}|^2}\bar{\mathbf{q}}\sum_{m=0}^{\infty}\frac{|\mathbf{q}|^{2m}}{m!} \\
&= \bar{\mathbf{q}},
\end{aligned}$$

and similarly,

$$\begin{aligned}
\langle\gamma_{\mathbf{q}}|A^2_{\bar{\mathbf{q}}}|\gamma_{\mathbf{q}}\rangle &= \bar{\mathbf{q}}^2, \\
\langle\gamma_{\mathbf{q}}|A_{\bar{\mathbf{q}}}A_{\mathbf{q}}|\gamma_{\mathbf{q}}\rangle &= \bar{\mathbf{q}}\mathbf{q} = |\mathbf{q}|^2, \\
\langle\gamma_{\mathbf{q}}|A_{\mathbf{q}}A_{\bar{\mathbf{q}}}|\gamma_{\mathbf{q}}\rangle &= 1+|\mathbf{q}|^2.
\end{aligned}$$

Using the above expectation values we can get the expectation values of Q and Q^2 as follows.

$$\begin{aligned}
\langle\gamma_{\mathbf{q}}|Q|\gamma_{\mathbf{q}}\rangle &= \frac{1}{\sqrt{2}}\langle\gamma_{\mathbf{q}}|A_{\mathbf{q}}+A_{\bar{\mathbf{q}}}|\gamma_{\mathbf{q}}\rangle \\
&= \frac{1}{\sqrt{2}}[\langle\gamma_{\mathbf{q}}|A_{\mathbf{q}}|\gamma_{\mathbf{q}}\rangle + \langle\gamma_{\mathbf{q}}|A_{\bar{\mathbf{q}}}|\gamma_{\mathbf{q}}\rangle] \\
&= \frac{1}{\sqrt{2}}(\mathbf{q}+\bar{\mathbf{q}}),
\end{aligned}$$

and hence

$$\langle\gamma_{\mathbf{q}}|Q|\gamma_{\mathbf{q}}\rangle^2 = \frac{1}{2}(\mathbf{q}^2+2|\mathbf{q}|^2+\bar{\mathbf{q}}^2).$$

Now for Q^2

$$\begin{aligned}
&\langle\gamma_{\mathbf{q}}|Q^2|\gamma_{\mathbf{q}}\rangle \\
&= \frac{1}{2}\langle\gamma_{\mathbf{q}}|A^2_{\mathbf{q}}+A_{\mathbf{q}}A_{\bar{\mathbf{q}}}+A_{\bar{\mathbf{q}}}A_{\mathbf{q}}A^2_{\bar{\mathbf{q}}}|\gamma_{\mathbf{q}}\rangle \\
&= \frac{1}{2}[\mathbf{q}^2+1+|\mathbf{q}|^2+|\mathbf{q}|^2+\bar{\mathbf{q}}^2] \\
&= \frac{1}{2}[\mathbf{q}^2+1+2|\mathbf{q}|^2+\bar{\mathbf{q}}^2].
\end{aligned}$$

Therefore the variance of Q becomes

$$\begin{aligned}
\langle\Delta Q\rangle^2 &= \langle\gamma_{\mathbf{q}}|Q^2|\gamma_{\mathbf{q}}\rangle - \langle\gamma_{\mathbf{q}}|Q|\gamma_{\mathbf{q}}\rangle^2 \\
&= 1/2.
\end{aligned}$$

That is,

$$\langle\Delta Q\rangle = \frac{1}{\sqrt{2}}.$$

For the momentum operator P, we have

$$\begin{aligned}
P|\gamma_{\mathbf{q}}\rangle &= \left(\frac{-I}{\sqrt{2}}[A_{\mathbf{q}}-A_{\bar{\mathbf{q}}}]\right)|\gamma_{\mathbf{q}}\rangle \\
&= ([A_{\mathbf{q}}-A_{\bar{\mathbf{q}}}]|\gamma_{\mathbf{q}}\rangle)\overline{\left(\frac{-I}{\sqrt{2}}\right)} \\
&= ([A_{\mathbf{q}}-A_{\bar{\mathbf{q}}}]|\gamma_{\mathbf{q}}\rangle)\left(\frac{I}{\sqrt{2}}\right).
\end{aligned}$$

Thereby we get

$$\begin{aligned}
\langle\gamma_{\mathbf{q}}|P|\gamma_{\mathbf{q}}\rangle &= \langle\gamma_{\mathbf{q}}|A_{\mathbf{q}}-A_{\bar{\mathbf{q}}}|\gamma_{\mathbf{q}}\rangle\frac{I}{\sqrt{2}} \\
&= [\langle\gamma_{\mathbf{q}}|A_{\mathbf{q}}|\gamma_{\mathbf{q}}\rangle - \langle\gamma_{\mathbf{q}}|A_{\bar{\mathbf{q}}}|\gamma_{\mathbf{q}}\rangle]\frac{I}{\sqrt{2}} \\
&= (\mathbf{q}-\bar{\mathbf{q}})\frac{I}{\sqrt{2}},
\end{aligned}$$

hence, as $I^2=-1$, we obtain

$$\langle\gamma_{\mathbf{q}}|P|\gamma_{\mathbf{q}}\rangle^2 = \frac{1}{2}(-\mathbf{q}^2+2|\mathbf{q}|^2-\bar{\mathbf{q}}^2).$$

Now for P^2

$$\begin{aligned}
&\langle\gamma_{\mathbf{q}}|P^2|\gamma_{\mathbf{q}}\rangle \\
&= -\frac{1}{2}\langle\gamma_{\mathbf{q}}|A^2_{\mathbf{q}}-A_{\mathbf{q}}A_{\bar{\mathbf{q}}}-A_{\bar{\mathbf{q}}}A_{\mathbf{q}}+A^2_{\bar{\mathbf{q}}}|\gamma_{\mathbf{q}}\rangle \\
&= -\frac{1}{2}[\mathbf{q}^2-1-|\mathbf{q}|^2-|\mathbf{q}|^2+\bar{\mathbf{q}}^2] \\
&= -\frac{1}{2}[\mathbf{q}^2-1-2|\mathbf{q}|^2+\bar{\mathbf{q}}^2].
\end{aligned}$$

Therefore the variance of P becomes

$$\begin{aligned}
\langle\Delta P\rangle^2 &= \langle\gamma_{\mathbf{q}}|P^2|\gamma_{\mathbf{q}}\rangle - \langle\gamma_{\mathbf{q}}|P|\gamma_{\mathbf{q}}\rangle^2 \\
&= 1/2.
\end{aligned}$$

That is,

$$\langle\Delta P\rangle = \frac{1}{\sqrt{2}}.$$

As the conclusion of the above, we have

$$\langle\Delta Q\rangle\langle\Delta P\rangle = \frac{1}{2}.$$

Further, since $[Q,P]=I\mathbb{I}_{\mathfrak{H}}$, we have

$$\begin{aligned}
[Q,P]|\gamma_{\mathbf{q}}\rangle &= (I\mathbb{I}_{\mathfrak{H}})|\gamma_{\mathbf{q}}\rangle = (\mathbb{I}_{\mathfrak{H}}|\gamma_{\mathbf{q}}\rangle)\bar{I} \\
&= |\gamma_{\mathbf{q}}\rangle(-I).
\end{aligned}$$

Therefore

$$\langle\gamma_{\mathbf{q}}|[Q,P]|\gamma_{\mathbf{q}}\rangle = \langle\gamma_{\mathbf{q}}|\gamma_{\mathbf{q}}\rangle(-I) = -I.$$

Hence

$$\frac{1}{2}|\langle[Q,P]\rangle| = \frac{1}{2}|-I| = \frac{1}{2}.$$

The above can be recapitulated in one line as

$$\langle\Delta Q\rangle\langle\Delta P\rangle = \frac{1}{2}|\langle[Q,P]\rangle| = \frac{1}{2}.$$

That is, the RQCS $|\gamma_{\mathbf{q}}\rangle$ saturate the Heisenberg uncertainty and, due to $[Q,P]=I\mathbb{I}_{\mathfrak{H}}$, the RQCS are minimum uncertainty states and are intelligent states too, which is in complete analogy with the canonical CS of CQM.

6. Conclusion

Using the general scheme of CS quantization the quaternion field is quantized in [1]. Using the annihilation operator, $A_{\mathbf{q}}$, and the creation operator, $A_{\overline{\mathbf{q}}}$, in [1] the momentum operator, P, and the position operator, Q, are obtained as self-adjoint operators in a quaternionic Hilbert space. For the RQCS, and for the operators P and Q, in this article, we have examined the Heisenberg uncertainty principle. In fact, as expected, the RQCS saturated the Heisenberg uncertainty, and thereby they formed a set of intelligent states. Further, since the operators P and Q satisfied the commutator relation $[Q,P]=I\mathbb{I}_{\mathfrak{H}}$, we have presented the RQCS as minimum uncertainty states. In conclusion, even though the noncommutativity of quaternions caused technical difficulties, in most part, the quantization procedure and the Heisenberg principle of quaternions followed its complex counterpart. As the quantization and the Heisenberg principle play an important role in complex quantum mechanics, the material presented in this manuscript can also play a vital role in the quaternionic quantum mechanics.

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